

A NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

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A NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

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Pettis¹ raised the question whether or not separability of the range space implies almost everywhere weak differentiability of Pettis integrals. Phillips² has given an example which answers this question in the negative. His construction is based on a sequence of orthogonal vectors in Hilbert space. We present here a different example of the same type of function. Our basic construction is that of a function defined to the space C . Using that function as a basis, we are able to give a specific construction of such a function defined to each member of a large class of Banach spaces.

1. Metric density properties of a non-dense perfect set. Let $B \subset [0, 1]$ be a non-dense perfect set of measure one-half, and let $\bar{B}$ be its complement. $\bar{B}$ may be constructed by taking the sum of a set of open intervals classified as follows:

- 1 interval of length $1/4$,
- 2 intervals each of length $1/16$,
- 4 intervals each of length $1/64$,
-
- 2^{n-1} intervals each of length $1/2^{2n}$,
-

We shall refer to the intervals of length $1/2^{2n}$ as *intervals of $\bar{B}$ of order n* . We shall assume that each interval of $\bar{B}$ of order n is the center portion of the space either between two intervals of $\bar{B}$ of lower order or between one such interval of $\bar{B}$ and an end point of the unit interval. These spaces we shall refer to as *gaps of order n* , and we shall denote such a gap by the symbol G_n . If $\bar{B}$ is constructed as noted above, then for each n , any two sets each of the form $G_n \cdot \bar{B}$ are congruent; hence we shall use G_n to denote a gap of order n , and we shall not find it necessary to specify which one.

The following three lemmas are now obvious.

1.1. LEMMA. $|\bar{B} \cdot G_n| = 1/2^{2n-1}$.

1.2. LEMMA. $|G_n| = 1/2^n + 1/2^{2n-1}$.

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¹ See [3, p. 303]. Numbers in brackets refer to the references cited at the end of the paper.

² See [4, p. 144].

1.3. DEFINITION. If I is any subinterval of $[0, 1]$, we define

$$\rho(I) = |\overline{B} \cdot I| / |I|.$$

1.4. LEMMA. $\rho(G_n) = 1/(1 + 2^{n-1})$.

The following important lemma demonstrates a lower bound on the density function ρ . This lower bound may tend to zero as $|I|$ tends to zero, but it is independent of the location of I .

1.5. LEMMA. If

$$|I| \geq |G_n| = 1/2^n + 1/2^{2n-1},$$

then

$$\rho(I) > 1/(2 + 2^{n-1}).$$

First suppose $|I| = |G_n|$. For all such I the G_n have minimum values for ρ ; for if $I = G_n$ and then is moved a little to the right or left, some points of B are excluded and only points of $\overline{B}$ are included—thus obviously increasing $\rho(I)$ —until the interval of $\overline{B}$ is covered. However, this interval of $\overline{B}$ is of order at most $n-1$ and hence has measure at least $1/2^{2n-2}$; thus so long as I contains this interval of $\overline{B}$,

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^n + 1/2^{2n-1}} = \frac{2}{1 + 2^{n-1}} = 2\rho(G_n).$$

If I is moved on beyond this interval of $\overline{B}$ the above argument applies again by considering the movement in the reverse direction. Comparison with Lemma 1.4 now shows that Lemma 1.5 is established in case $|I| = |G_n|$.

Consider now the effect of increasing $|I|$. If $|I| = |G_n|$ and I contains no interval of $\overline{B}$ of order less than or equal to $n-1$, then one end point of I (let us assume it is the right-hand one) lies in the closure of such an interval of $\overline{B}$. Thus any small extension of I to the right will (until the interval of $\overline{B}$ is covered) add to I only points of $\overline{B}$, thus obviously increasing $\rho(I)$. If $I = G_n$ at the start, a small extension to the left will have the same effect. Otherwise an extension to the left may be regarded as a translation to the left and a subsequent extension to the right, and these cases have already been discussed. In case I contains an interval of $\overline{B}$ of order not greater than $n-1$, we have $|\overline{B} \cdot I| > 1/2^{2n-2}$; thus if $|I| \leq |G_{n-1}| = 1/2^{n-1} + 1/2^{2n-3}$,

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^{n-1} + 1/2^{2n-3}} = \frac{1}{2 + 2^{n-1}}.$$

If $|I| > |G_{n-1}|$, the above argument may be repeated with $n-1$ sub-

stituted for n , thereby obtaining an even larger lower bound for $\rho(I)$. This proves Lemma 1.5 for all cases.

1.6. LEMMA. For all $I \subset [0, 1]$,

$$\rho(I) > |\bar{B} \cdot I|^{1/2} / 2^{5/2}.$$

We shall prove Lemma 1.6 by showing that, for each n , the required inequality holds for

$$1/2^{2n+1} < |\bar{B} \cdot I| \leq 1/2^{2n-1}.$$

This will cover all possibilities. For $|\bar{B} \cdot I|$ in this range, we consider first the case $|I| < |G_n| = 1/2^n + 1/2^{2n-1}$. In this case

$$\begin{aligned} \frac{1}{\rho(I)} &= \frac{|I|}{|\bar{B} \cdot I|} < \frac{|I|}{1/2^{2n+1}} < \frac{1/2^n + 1/2^{2n-1}}{1/2^{2n+1}} = 4 + 2^{n+1} \\ &= 4 + 2^{3/2} [1/2^{2n-1}]^{-1/2} \leq 4 + 2^{3/2} |\bar{B} \cdot I|^{-1/2}. \end{aligned}$$

Considering now the case $|I| \geq |G_n|$ (and assuming $|\bar{B} \cdot I|$ still in the same range) we have, using Lemma 1.5 and the above inequalities,

$$1/\rho(I) < 2 + 2^{n-1} < 4 + 2^{n+1} \leq 4 + 2^{3/2} |\bar{B} \cdot I|^{-1/2}.$$

Now for all $I \subset [0, 1]$, $|\bar{B} \cdot I| \leq 1/2$; hence $|\bar{B} \cdot I|^{-1/2} \geq 2^{1/2}$; hence $4 \leq 2^{3/2} |\bar{B} \cdot I|^{-1/2}$. Combining this with the above results, we have

$$1/\rho(I) < 4 + 2^{3/2} |\bar{B} \cdot I|^{-1/2} \leq 2^{5/2} |\bar{B} \cdot I|^{-1/2},$$

whence

$$\rho(I) > |\bar{B} \cdot I|^{1/2} / 2^{5/2}.$$

2. An approximately continuous function whose integral is non-differentiable. For each $t \in B$ we define the function $f_t(x)$ for $x \in [0, 1]$ as follows

$$f_t(x) = \begin{cases} 0 & \text{for } x \leq t \text{ or } x \in B, \\ |\bar{B} \cdot [t, x]|^{-3/4} & \text{for } x > t \text{ and } x \in \bar{B}. \end{cases}$$

2.1. THEOREM. For each $t \in B$, $f_t(x)$ is an integrable function of x , and for $x_2 \geq x_1 \geq t$,

$$\int_{x_1}^{x_2} f_t(x) dx = 4(|\bar{B} \cdot [t, x_2]|^{1/4} - |\bar{B} \cdot [t, x_1]|^{1/4}).$$

Let $z = |\bar{B} \cdot [t, x]|$. Since the intervals of $\bar{B}$ are dense in $[0, 1]$, this defines z as a strictly monotone function of x ; hence x is a single-valued function of z , and we may write $f_t[x(z)]$. Now for $x > t$ and $x \in \bar{B}$, $dx = dz$; thus the function $z(x)$ is measure preserving over $\bar{B}$

$\cdot [t, x]$ and hence maps $B \cdot [t, x]$ into a set of measure zero. Therefore, for almost all z , $f_t[x(z)] = z^{-3/4}$; and $dx = dz$ except where $f_t(x) = 0$; so

$$\int_{x_1}^{x_2} f_t(x) dx = \int_{z_1}^{z_2} z^{-3/4} dz = 4(z_2^{1/4} - z_1^{1/4}).$$

2.2. THEOREM. For $t \in B$, the function

$$F_t(x) = \int_t^x f_t(u) du$$

is not differentiable with respect to x at $x = t$.

Again letting $z = |\bar{B} \cdot [t, x]|$, and using Theorem 2.1 and Lemma 1.6, we have

$$\begin{aligned} \frac{F_t(x)}{x-t} &= \frac{4z^{1/4}}{x-t} = 4z^{-3/4} \left(\frac{z}{x-t} \right) = 4z^{-3/4} \rho([t, x]) \\ &> 4z^{-3/4} (z^{1/2}/2^{5/2}) = (4z)^{-1/4}. \end{aligned}$$

Thus

$$\limsup_{x \rightarrow t} \frac{F_t(x)}{x-t} \geq \lim_{z \rightarrow 0} (4z)^{-1/4} = \infty.$$

In the next section we shall make further use of the functions $f_t(x)$ and their properties as shown in Theorems 2.1 and 2.2. We might note here, however, that $f_t(x)$ is approximately continuous at t provided $\bar{B}$ has metric density zero at t . This is true for almost all t in B ; hence for such t , $f_t(x)$ furnishes a specific example of an approximately continuous function whose integral is not differentiable.

3. A Pettis integral in the space C which is not almost everywhere weakly differentiable. We shall here define a function $\phi(x)$ from $[0, 1]$ to the space C . Our notation will be as follows: For each $x \in [0, 1]$, $\phi(x)$ stands for a continuous function on $[0, 1]$; we denote this continuous function by $\phi_x(t)$. We shall define the functions $\phi_x(t)$ by defining a function $\phi(x, t)$ over the unit square and setting $\phi_x(t) = \phi(x, t)$. We first define $\phi(x, t)$ over a portion of the unit square as follows:

$$\phi(x, t) = \begin{cases} 0 & \text{for } x \in B, \\ f_t(x) & \text{for } t \in B. \end{cases}$$

Since $f_t(x) = 0$ for $x \in B$, these statements are consistent.

3.1. LEMMA. For a fixed x , $\phi(x, t)$ is continuous in t over B .

This statement follows immediately from the fact that if one end

point of I is fixed, $|\bar{B} \cdot I|^{-3/4}$ is a continuous function of the other end point over any set such that $|I|$ is bounded away from zero. This latter restriction causes no difficulties here. If $x \in B$, $\phi(x, t) = 0$; if $x \in \bar{B}$, $\text{dist}(x, B) > 0$.

We now continue the definition of $\phi(x, t)$. For each $x \in \bar{B}$, let $\phi(x, t)$ be continued linearly over each interval of the set $t \in \bar{B}$. This completes the definition of $\phi(x, t)$ over the entire unit square, and it is clear that for each x , $\phi(x, t)$ is continuous in t over $[0, 1]$.

3.2. THEOREM. $\phi(x)$ is integrable in the sense of Pettis. For each measurable set $E \subset [0, 1]$, its integral over E is the function

$$\Phi_E(t) = \int_E \phi(u, t) du.$$

We show this by considering the functions $\phi^{(n)}(x)$ whose values are the continuous functions $\phi_x^{(n)}(t) = \phi^{(n)}(x, t)$ where

$$\phi^{(n)}(x, t) = \begin{cases} \phi(x, t) & \text{for } \phi(x, t) \leq n, \\ n & \text{for } \phi(x, t) > n. \end{cases}$$

It is easily seen that for each $t \in [0, 1]$, each $\phi^{(n)}(x, t)$ is bounded and continuous in x over $\bar{B}$. Thus³ each $\phi^{(n)}(x)$ is weakly continuous over $\bar{B}$. Since $\phi^{(n)}(x) = \theta$ for $x \in B$, it is clear that each $\phi^{(n)}(x)$ is weakly measurable. Since C is a separable space, it follows⁴ that each $\phi^{(n)}(x)$ is measurable. Now each $\phi^{(n)}(x)$ is bounded, hence Bochner integrable, hence Pettis integrable, therefore integrable with respect to each of the linear functionals $\gamma_i[\phi^{(n)}(x)] = \phi_x^{(n)}(t)$; thus

$$\Phi^{(n)}(E) = \int_E \phi^{(n)}(u) du$$

is the continuous⁵ function

$$\Phi_E^{(n)}(t) = \int_E \phi^{(n)}(u, t) du.$$

Clearly for each x and each t ,

$$\lim_{n \rightarrow \infty} \phi_x^{(n)}(t) = \phi_x(t);$$

³ See [1, p. 224, Theorem 8].

⁴ See [3, Theorem 1.1].

⁵ An independent proof of continuity of $\Phi_E^{(n)}(t)$ is unnecessary. Since $\phi^{(n)}(x)$ is Pettis integrable, it is integrable to an element of C ; and the set $\{\gamma_i\}$ of linear functionals defines this element uniquely.

furthermore this approximation is monotone in n . Thus $\|\phi^{(n)}(x) - \phi(x)\|$ is bounded in n for each x ; hence⁶ $\phi^{(n)}(x) \rightarrow \phi(x)$ weakly for each x . It now follows⁷ that $\phi(x)$ is Pettis integrable provided the sequence $\{\Phi^{(n)}(E)\}$ converges with respect to the norm in C ; that is, provided $\{\Phi_E^{(n)}(t)\}$ converges uniformly in t . We shall complete the proof of Theorem 3.2 by showing that for each measurable $E \subset [0, 1]$

$$\Phi_E(t) = \int_E \phi(u, t) du$$

exists for each t and that this function is the uniform limit of the sequence $\{\Phi_E^{(n)}(t)\}$.

To show that $\Phi_E(t)$ exists is trivial. For $t \in B$, this follows from Theorem 2.1. For each x , $\phi(x, t)$ is extended linearly over each interval of $t \in \bar{B}$; hence for $t \in \bar{B}$, $\phi(x, t) \leq \phi(x, t_1) + \phi(x, t_2)$ where t_1 and t_2 are each in B . This completes the proof of integrability.

Now with each $t \in [0, 1]$ we associate two numbers t_1 and t_2 as follows: t_1 is the greatest number such that $t_1 \in B$ and $t_1 \leq t$; t_2 is the smallest number such that $t_2 \in B$ and $t_2 \geq t$. Geometrically this means that if $t \in B$, $t_1 = t = t_2$, while if $t \in \bar{B}$, t_1 and t_2 are the left and right points respectively of the interval of $\bar{B}$ in which t is located.

Now for $t_1 \leq x < t_2$, $\phi(x, t) \leq \phi(x, t_1)$ while for $x \geq t_2$, $\phi(x, t) \leq \phi(x, t_2)$. Thus for any given $t \in [0, 1]$, it is possible to have $\phi(x, t) > n$ only for those values of x for which either

$$t_1 \leq x < t_2 \quad \text{and} \quad |\bar{B} \cdot [t_1, x]| < n^{-4/3}$$

or

$$x \geq t_2 \quad \text{and} \quad |\bar{B} \cdot [t_2, x]| < n^{-4/3}.$$

Outside these two intervals $\phi(x, t) - \phi^{(n)}(x, t) = 0$; hence if we denote these intervals by I_1 and I_2 , we have

$$\begin{aligned} \int_E [\phi(x, t) - \phi^{(n)}(x, t)] dx &\leq \int_{I_1} \phi(x, t) dx + \int_{I_2} \phi(x, t) dx \\ &\leq \int_{I_1} \phi(x, t_1) dx + \int_{I_2} \phi(x, t_2) dx \\ &< 2 \int_0^{n^{-4/3}} z^{-3/4} dz = 8n^{-1/3}. \end{aligned}$$

⁶ See [1, p. 224, Theorem 8].

⁷ See [3, Theorem 4.1].

Thus, clearly, $\lim_{n \rightarrow \infty} \Phi_E^{(n)}(t) = \Phi_E(t)$ uniformly in t . This completes the proof of Theorem 3.2.

3.3. THEOREM. *If $x_0 \in B$, $\Phi(x) = \int_0^x \phi(u) du$ is not weakly differentiable at x_0 .*

By Theorem 2.2, it fails to be differentiable at x_0 with respect to the linear functional $\gamma_{x_0}[\Phi(x)] = \Phi_x(x_0)$.

4. **Extension to other spaces of continuous functions.** The function $\phi(x)$ of §3 may be used as the basis for the construction of a large set of examples as follows:

4.1. THEOREM. *If Ω is a compact metric space containing non-denumerably many points and if $C(\Omega)$ is the Banach space of all continuous functionals on Ω , then there is a function $\psi(x)$ from the unit interval to $C(\Omega)$ such that $\psi(x)$ is Pettis integrable but $\Psi(E) = \int_E \psi(x) dx$ fails to be weakly differentiable on a set of positive measure.*

Since Ω is non-denumerable, it contains a perfect set. This perfect set is a complete metric space which is dense in itself and hence contains a homeomorph Π of the Cantor set B .⁸

Let $B = h(\Pi)$ be the homeomorphic mapping of Π into B . Then $h(\omega)$ is a continuous function defined over Π , assuming values between 0 and 1, and assuming for some $\omega \in \Pi$ each value in the set B . Let $H(\omega)$ be a continuous extension⁹ of $h(\omega)$ over the whole of Ω with $0 \leq H(\omega) \leq 1$.

Now for each $t \in [0, 1]$ we define

$$K(t) = \bigcup_{\omega} \{H(\omega) = t\}.$$

It should be noted that although for some t , $K(t)$ may be vacuous, for each $t \in B$, $K(t)$ contains at least one point.

Referring back to the functions $\phi_x(t)$ of §3, we now define

$$\psi_x(\omega) = \phi_x(t) \quad \text{for } \omega \in K(t).$$

It follows from the continuity in t of each function $\phi_x(t)$ and from the continuity of $H(\omega)$ that for each $x \in [0, 1]$, $\psi_x(\omega)$ is continuous over Ω . For each $x \in [0, 1]$ we now let $\psi(x)$ be the element $\psi_x(\omega)$ of $C(\Omega)$.

⁸ See [2, p. 228]. The author is indebted to the referee for the suggestion that non-denumerability of Ω is sufficient to insure the existence of Π .

⁹ See [2, p. 211]. In connection with our remark in the introduction that we have a specific construction applicable to the more general spaces, it should be noted that this extension theorem is not merely an existence proof. A definite formula for the extension is given.

That $\psi(x)$ has the required properties may be seen as follows: To show integrability, we note that for each $\omega \in \Omega$, $\psi_x(\omega)$ is identical (as a function of x) with $\phi_x(t)$ for some $t \in [0, 1]$. Then noting that Banach's criterion for weak convergence¹⁰ applies to the space $C(\Omega)$, we apply the proof of Theorem 3.2. To show non-differentiability, we note that for each $t \in B$ there is an $\omega \in \Omega$ such that $\psi_x(\omega) = \phi_x(t)$ for all $x \in [0, 1]$. The proof of Theorem 3.3 then applies.

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¹⁰ This is used in the proof of Theorem 3.2. See footnotes 3 and 6.

